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Geometry and the Low-Energy Theorem in $N = 1$ Supersymmetric Theories

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Abstract

We investigate geometrical structures and low-energy theorems of $N = 1$ supersymmetric nonlinear sigma models in four dimensions. When a global symmetry spontaneously breaks down to its subgroup, the low-energy effective Lagrangian of massless particles is described by a supersymmetric nonlinear sigma model whose target manifold is parametrized by Nambu-Goldstone (NG) bosons and quasi-NG (QNG) bosons. The unbroken symmetry changes at each point in the target manifold and some QNG bosons change to NG bosons when unbroken symmetry become smaller. The QNG-NG change and their interpretation is shown in a simple example, the $O(N)$ model. We investigate low-energy theorems at general points.

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1 Introduction

In non-supersymmetric theories, the Nambu-Goldstone (NG) theorem tells us that there appear as many massless NG bosons as the number of broken generators, namely dim (G/H), when a global symmetry G spontaneously breaks down to its subgroup H. The NG bosons parameterize a vacuum degeneracy which has one-to-one correspondence with the freedom of the embedding H into G. The effective Lagrangian of massless bosons can be expanded by the number of space-time derivatives, and the leading term, with two derivatives, is described by nonlinear sigma models on target manifolds G/H, whose coordinates are parametrized by NG bosons. On this manifold, the unbroken symmetry H is realized linearly, while the broken symmetry G is realized nonlinearly by NG bosons [1]. For non-supersymmetric cases, low-energy theorems tell us that low-energy scattering amplitudes of NG bosons are determined solely by the symmetries G and H, and do not depend on details of the underlying theory (for a review, see Ref. [2]). The effective Lagrangian reproduces these low-energy theorems.

In supersymmetric theories, there appear additional massless bosons called quasi-NG (QNG) bosons [3] (and their fermionic superpartners).[1] Leading terms of massless effective Lagrangian are described by $N = 1$ supersymmetric nonlinear sigma models (for example see Ref. [4]). Target manifolds of $N = 1$ nonlinear sigma models are Kähler manifolds [5]: A manifold whose metric is given by a Kähler potential $K(\varphi, \varphi^*)$

$$g_{ij^*}(\varphi, \varphi^*) = \frac{\partial^2 K(\varphi, \varphi^*)}{\partial \varphi^i \partial \varphi^{*j}},$$

is called a Kähler manifold. $\varphi(x)$ is a complex scalar component of a chiral superfield. NG and QNG bosons are coordinates of a complex coset manifold $G^{\mathbf{C}}/\hat{H}$, where $G^{\mathbf{C}}$ is the complexification of G and $\hat{H}$ is the complex subgroup often larger than $H^{\mathbf{C}}$, the complexification of H. Kähler potentials of $G^{\mathbf{C}}/\hat{H}$ have been constructed by Bando, Kuramoto, Maskawa and Uehara (BKMU) [6] (for a review, see Ref. [7]). If $\hat{H} = H^{\mathbf{C}}$, the number of QNG bosons is the same as that of NG bosons, and nonlinear realizations in these cases are called "maximal realizations" or "fully-doubled realizations". On the other hand, if $\hat{H}(\supset H^{\mathbf{C}})$ becomes larger, the number of QNG bosons decreases. If there is no QNG boson, realizations are called "pure realizations", and studied extensively [8]. Pure realizations cannot be obtained as the low-energy limit of underlying linear theories since there remains at least one QNG boson [9, 10, 11]. If there is, however, gauge symmetry, it is possible to absorb pairs of a QNG and a NG bosons by the supersymmetric Higgs mechanism. Hence pure realizations are in some cases obtained as low-energy theories of gauged linear sigma models [7, 12].

To investigate low-energy theories with supersymmetry, it is important to understand geometric structures of supersymmetric sigma models. In the cases of pure

[1] Fermions, called the QNG fermions, would be interesting particles, when we regard quarks and leptons as QNG fermions. But we do not discuss QNG fermions in this paper.

realizations, the geometry of the target space is well understood, because Kähler potentials are uniquely determined by the metric of G/H. When there are QNG bosons, however, the coset space, G/H where NG bosons reside, is a subspace of the target space. Since the metric in directions along QNG bosons is not determined by the geometry of its subspace G/H, the effective Lagrangian is not unique in this case and depends on an arbitrary function of many G-invariant variables [11, 13]. When there are many G-invariant variables, it is complicated to study geometric structures of target spaces in general. In this paper, we investigate the $O(N)$ model whose Kähler potential contains an arbitrary function of a single variable, but generalizations to other models are straightforward.

This paper is organized as follows. We review our previous results [14] in the rest of this section. The low-energy theorems at the symmetric points are explained. In Sect. 2, we study non-symmetric points where unbroken symmetry H is reduced to a smaller group H'. In Sect. 3, to investigate the geometrical structure of the supersymmetric nonlinear sigma model with $O(N)$ symmetry, we show explicitly how the different compact homogeneous manifolds G/H and G/H' are embedded in the full target manifold $G^{\mathbf{C}}/\hat{H}$ by using the method of Shore [15]. We see how some QNG bosons change to NG bosons at the non-symmetric points. In Sect. 4, we derive the low-energy theorems of NG and QNG bosons at the general points of the target manifold when the Kähler potential is the simplest one. Sect. 5 is devoted to conclusion and discussion. In Appendix A, we explain the Kähler normal coordinate which is used to calculate the low-energy theorems. In Appendix B, some geometric quantities are calculated for the most general $O(N)$-invariant model.

The general low-energy effective Lagrangian of massless bosons $\phi^\alpha(x)$ is a nonlinear sigma model whose target manifold has the metric $g_{\alpha\beta}(\phi)$,

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}(\phi)\partial_\mu\phi^\alpha\partial^\mu\phi^\beta. \tag{1.1}$$

Low-energy scattering amplitudes are unchanged by a field redefinition, which is a general coordinate transformation in the target manifold. By expanding this in the Riemann normal coordinate ϕ^i [16] up to the forth order, and regarding the fourth order terms as interaction terms $\mathcal{L}_{\rm int}$, low-energy two-body scattering amplitudes of the massless bosons ϕ^i (with momenta p_i)

$$\begin{aligned}&\left\langle \phi^k(p_k), \phi^l(p_l)|i\mathcal{L}_{int}|\phi^i(p_i), \phi^j(p_j)\right\rangle \\ &= i(2\pi)^4\delta^{(4)}(p_k+p_l-p_i-p_j)\mathcal{M}(\phi^i(p_i), \phi^j(p_j) \to \phi^k(p_k), \phi^l(p_l))\end{aligned} \tag{1.2}$$

can be calculated by summing up all the tree graphs[2], given by

$$\mathcal{M}(\phi^i, \phi^j \to \phi^k, \phi^l) = -\frac{1}{3f_\pi^4}[(s-u)R_{kijl} + (u-t)R_{ijkl} + (t-s)R_{kjli}]. \tag{1.3}$$

[2] To calculate the next-to leading order $\mathcal{O}(p^4)$, we need to sum up one-loop graphs of the leading term and tree graphs of the four derivative terms, with obeying Weinberg's counting theorem.

Here f_π is the decay constant of the NG bosons (pions), R_{ijkl} is the curvature tensor of the target manifold, and we have defined the Mandelstam's variables by

$$
\begin{aligned}
s &\stackrel{\text{def}}{=} (p_i + p_j)^2 = +2p_i \cdot p_j = +2p_l \cdot p_k, \\
t &\stackrel{\text{def}}{=} (p_i - p_k)^2 = -2p_i \cdot p_k = -2p_l \cdot p_j, \\
u &\stackrel{\text{def}}{=} (p_i - p_l)^2 = -2p_i \cdot p_l = -2p_j \cdot p_l. \qquad (1.4)
\end{aligned}
$$

We consider cases that a global symmetry G spontaneously breaks down to its subgroup H. We express broken and unbroken generators by [3]

$$X_i \in \mathcal{G} - \mathcal{H}, \qquad H_a \in \mathcal{H}, \qquad (T_A \in \mathcal{G}). \qquad (1.5)$$

In the cases of symmetric spaces G/H (in which there is a symmetry $X_i \to -X_i$, $H_a \to H_a$), the curvature tensor can be calculated by using structure constants of G, ${f_{AB}}^C$, to yield

$$R_{ijkl} = f_\pi^2 \, {f_{ij}}^a f_{akl}, \qquad (1.6)$$

and the low-energy theorems become

$$\mathcal{M}(\phi^i, \phi^j \to \phi^k, \phi^l) = -\frac{1}{3f_\pi^2}[(s-u){f_{ki}}^a f_{ajl} + (u-t){f_{kl}}^a f_{aij} + (t-s){f_{kj}}^a f_{ali}]. \qquad (1.7)$$

In $N = 1$ supersymmetric theories, the low-energy effective Lagrangian of massless chiral superfields $\Phi^i(x, \theta, \bar\theta) = \varphi^i(x) + \sqrt{2}\theta\psi^i(x) + \theta\theta F^i(x)$ (where φ^i, ψ^i and F^i are complex scalar fields, Weyl fermions, auxiliary scalar fields, respectively.) is a supersymmetric nonlinear sigma model [22],

$$
\begin{aligned}
\mathcal{L} &= \int d^2\theta d^2\bar\theta K(\Phi, \Phi^\dagger) \\
&= g_{ij^*}(\varphi, \varphi^*)\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + i g_{ij^*}\bar\psi^j\bar\sigma^\mu(\partial_\mu\psi^i + {\Gamma^i}_{lk}\partial_\mu\varphi^l\psi^k) \\
&\quad + \frac{1}{4}R_{ij^*kl^*}\psi^i\psi^k\bar\psi^j\bar\psi^l. \qquad (1.8)
\end{aligned}
$$

Here the metric tensor is calculated by the Kähler potential as

$$g_{ij^*}(\varphi, \varphi^*) = \partial_i\partial_{j^*}K(\varphi, \varphi^*), \qquad (1.9)$$

and $R_{ij^*kl^*}$ and ${\Gamma^i}_{lk}$ are the complex curvature and the connection, respectively. In Eq. (1.8), the auxiliary fields F^i have been eliminated by using their equations of motion. The massless chiral NG superfields appear when the global symmetry G spontaneously breaks down to its subgroup H with preserving $N = 1$ supersymmetry. We denote complex broken and unbroken generators as [4]

$$Z_R \in \mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}, \qquad K_M \in \hat{\mathcal{H}}. \qquad (1.10)$$

[3] The Lie algebras of the groups G and H are denoted by $\mathcal{G}$ and $\mathcal{H}$, respectively.

[4] Complex generators are complex linear combinations of Hermitian generators of $\mathcal{G}$. We use indices R, S, T and L, M, N for *complex* broken and unbroken generators, respectively.

Their commutation relations are

$$[K_M, K_N] = if_{MN}{}^L K_L, \quad [Z_R, K_M] = if_{RM}{}^S Z_S, \quad [Z_R, Z_S] = if_{RS}{}^M K_M, \tag{1.11}$$

where we have assumed the existence of an automorphism

$$Z_R \to -Z_R, \qquad K_M \to K_M. \tag{1.12}$$

The target manifold (which is a $G^{\mathbf{C}}$-orbit of vacuum vector $\vec{v}$) is a complex coset manifold $G^{\mathbf{C}}/\hat{H}$, and its representative is

$$\xi(\Phi) = \mathrm{e}^{i\Phi\cdot Z} \in G^{\mathbf{C}}/\hat{H}, \qquad \Phi \cdot Z = \sum_{i=1}^{N_\Phi} \Phi^i Z_R \delta_i^R. \tag{1.13}$$

Here $\Phi^i(x, \theta, \bar{\theta})$ are the NG chiral superfields and N_Φ is a number of Φ^i. The left action of G on the coset representative is

$$\xi \xrightarrow{g} \xi' = g\xi\hat{h}^{-1}(g, \xi), \qquad g \in G, \tag{1.14}$$

where $\hat{h}(g, \xi) \in \hat{H}$ is called an $\hat{H}$-compensator.

It is known that, when a vacuum vector $\vec{v}$ is in the real representation of G, or when G/H is a symmetric space, only maximal realizations are possible [9]. So we discuss maximal realizations, where there appear the same numbers of NG and QNG bosons (at a symmetric point defined below). The low-energy effective Kähler potential can be written as [9, 6, 15, 11]

$$K(\Phi, \Phi^\dagger) = f(\vec{v}^{\,\dagger}\xi^\dagger(\Phi^\dagger)\xi(\Phi)\vec{v}) , \tag{1.15}$$

where f is an *arbitrary* function, which cannot be determined by symmetry.[5] This arbitrariness is a characteristic feature of non-pure realizations. Note that this Kähler potential is G-invariant by Eq. (1.14) but *not* $G^{\mathbf{C}}$-invariant: A G-action is a general coordinate transformation preserving the metric (the Kähler potential), while a $G^{\mathbf{C}}$-action does not preserve the metric. This fact has an important consequence: The symmetry of the action is still the compact real group G, although the target space is a $G^{\mathbf{C}}$-orbit of vacuum vector $\vec{v}$.

A holomorphic vielbein E_i^R and a canonical $\hat{H}$-connection W_i^M can be read as coefficients of broken and unbroken elements of the Maurer-Cartan 1-form

$$\frac{1}{i}\xi(\varphi)^{-1}d\xi(\varphi) = (E_i^R(\varphi)Z_R + W_i^M(\varphi)K_M)d\varphi^i. \tag{1.16}$$

We define *symmetric points* by points with the largest unbroken symmetry. (As seen in the next section, there exist points with smaller unbroken symmetry.) We can take a coordinate system φ on $G^{\mathbf{C}}/\hat{H}$ so that the origin of φ is a symmetric point.

[5] If there are some G-invariants, a Kähler potential can be written as an arbitrary function of such invariants.

Then at a symmetric point $\varphi = 0$, the vielbein and the $\hat{H}$ connection take the form of

$$E_i^R|_{\varphi=0} = \delta_i^R, \qquad W_i^M|_{\varphi=0} = 0, \tag{1.17}$$

respectively, and differentiations of the vielbein with respect to coordinates at the point are

$$\partial_j E_i^R|_{\varphi=0} = 0 \ . \tag{1.18}$$

From Eqs. (1.16) to (1.18), We can calculate the curvature tensor at the symmetric point, given by

$$R_{ij^*kl^*} = f_1(\vec{v}^\dagger Z_S^\dagger Z_V^\dagger Z_U Z_R \vec{v})\delta_i^R(\delta_j^S)^*\delta_k^U(\delta_l^V)^* + g^2(\delta_{ik}\delta_{j^*l^*} + \delta_{ij^*}\delta_{kl^*} + \delta_{il^*}\delta_{kj^*}), \tag{1.19}$$

where we have defined $v^2 \stackrel{\text{def}}{=} \vec{v}^\dagger \vec{v}$, $f_1 \stackrel{\text{def}}{=} f'(v^2)$, $f_2 \stackrel{\text{def}}{=} f''(v^2)$ etc. and a constant g by

$$g^2 \stackrel{\text{def}}{=} f_2\, v^2. \tag{1.20}$$

We express complex scalar fields by $\varphi^i(x) = A^i(x) + iB^i(x)$, where $A^i(x)$ and $B^i(x)$ are real scalar fields. In maximal realization cases, $A^i(x)$ and $B^i(x)$ are NG and QNG bosons, respectively. In the real basis of A^i and B^i, the Kähler condition on the curvature tensor becomes

$$\begin{cases} R_{A^iA^jA^kA^l} = R_{B^iB^jB^kB^l} = R_{A^iA^jB^kB^l} = R_{B^iB^jA^kA^l}, \\ R_{B^iA^jB^kA^l} = R_{A^iB^jA^kB^l} = -R_{B^iA^jA^kB^l} = -R_{A^iB^jB^kA^l}, \\ R_{B^iA^jA^kA^l} = -R_{A^iB^jA^kA^l} = -R_{A^iB^jB^kB^l} = R_{B^iA^jB^kB^l}, \\ R_{A^iA^jB^kA^l} = -R_{A^iA^jA^kB^l} = -R_{B^iB^jA^kB^l} = R_{B^iB^jB^kA^l}. \end{cases} \tag{1.21}$$

We can calculate real components of the curvature tensor (at symmetric points), which are directly related with low-energy scattering amplitudes, given by

$$\begin{aligned} R_{A^iA^jA^kA^l} &= f_\pi^2 f_{RS}{}^M f_{MUV}\delta_i^R\delta_j^S\delta_k^U\delta_l^V, \\ R_{B^iA^jB^kA^l} &= -f_1\vec{v}^{\,\dagger}(Z_S\{Z_V, Z_U\}Z_R + Z_R\{Z_V, Z_U\}Z_S)\vec{v}\ \delta_i^R\delta_j^S\delta_k^U\delta_l^V \\ &\quad -4g^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl}), \\ R_{B^iA^jA^kA^l} &= R_{A^iA^jB^kA^l} = 0, \end{aligned} \tag{1.22}$$

where we have defined $f_\pi^2 \stackrel{\text{def}}{=} 2f_1 v^2$. We thus obtain low-energy ($\mathcal{O}(p^2)$) scattering amplitudes of the NG and QNG bosons, by substituting Eqs. (1.22) and (1.21) to Eq. (1.3). We conclude that, at a symmetric point, there exist low-energy theorems of amplitudes which include only the NG bosons, where higher derivatives of the arbitrary function cancel out, and they coincide with scattering amplitudes among NG bosons in non-supersymmetric theories on a symmetric space G/H (see Eq. (1.7)). Amplitudes among only the QNG bosons coincide with those of the corresponding NG bosons by the Kähler conditions (1.21). Amplitudes for even number of the NG and QNG bosons depend on the second derivative of the arbitrary function.

We would like to generalize these results to low-energy theorems at general points. At non-symmetric points, some of the QNG bosons turn to NG bosons, corresponding to the fewer unbroken symmetry.

2 Non-symmetric points and supersymmetric vacuum alignment

In the last section we have discussed low-energy theorems of NG and QNG bosons at a symmetric point. In supersymmetric low-energy theories, there can exist points with smaller unbroken symmertry in the same vacuum manifold, as a result of the supersymmetric vacuum alignment. In this section we discuss how this phenomenon occurs.

2.1 Non-symmetric points

In maximal realizations, the number of the QNG bosons is equal to that of QNG bosons at symmetric points. If we leave from the symmetric point by a G-action as $\vec{v}' = g\vec{v}$ $(g \in G)$, they are also symmetric points and the unbroken symmetry remains unchanged : $H' = gHg^{-1} \simeq H$. The Kähler potential, the metric and the curvature tensor do not change and the low-energy theorems do not change either. All of them are equivalent vacua. The full target manifold is, however, constructed by $G^{\mathbf{C}}$-actions on $\vec{v}$. If we move to another vacua by a $G^{\mathbf{C}}$-action, the unbroken symmetry H varies depending on the choice of vacuum. A $G^{\mathbf{C}}$-action on the symmetric point $\vec{v}$ is

$$\vec{v}' = g_0\vec{v}, \qquad g_0 \in G^{\mathbf{C}} \ . \tag{2.1}$$

Complex unbroken subgroups at $\vec{v}$ and $\vec{v}'$, defined by

$$\hat{H}\vec{v} = \vec{v}, \qquad \hat{H}'\vec{v}' = \vec{v}' \ , \tag{2.2}$$

are related by

$$\hat{H}' = g_0\hat{H}{g_0}^{-1} \simeq \hat{H} : \qquad {K_M}' = g_0 K_M {g_0}^{-1} \ . \tag{2.3}$$

In this sense, the complex unbroken generators are equivalent at any point on the manifold . Complex broken generators are also related by

$$\mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}' = g_0(\mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}){g_0}^{-1} : \qquad {Z_R}' = g_0 Z_R {g_0}^{-1} \ . \tag{2.4}$$

Since $G^{\mathbf{C}}$-orbits of $\vec{v}$ and $\vec{v}'$ are homeomorphic to each other

$$G^{\mathbf{C}}/\hat{H} \simeq G^{\mathbf{C}}/\hat{H}' \ , \tag{2.5}$$

the transformation of $G^{\mathbf{C}}$ is just an automorphism on the target manifold. To be precise, (bosonic part of) the representatives of the complex cosets

$$\xi = \exp(i\varphi^R Z_R) \in G^{\mathbf{C}}/\hat{H}, \qquad \xi' = \exp(i\varphi'^R {Z_R}') \in G^{\mathbf{C}}/\hat{H}' \tag{2.6}$$

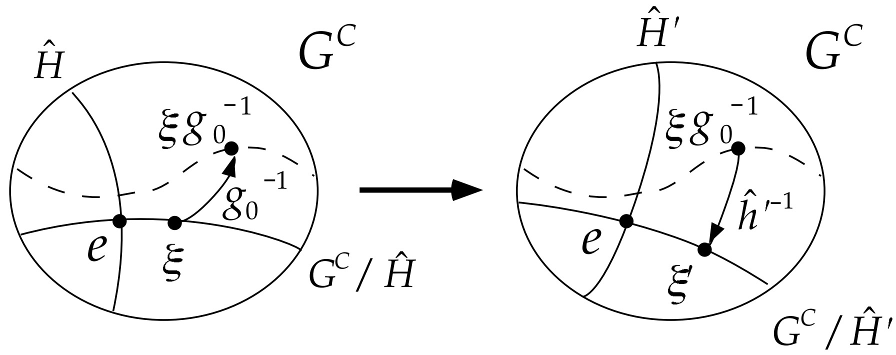

Figure 1

The complex unbroken symmetry $\hat{H}$ is transformed to $\hat{H}'$ in $G^{\mathbf{C}}$. The coset representatives of $G^{\mathbf{C}}/\hat{H}$, ξ, which is expressed by a horizontal curve, are transformed to a broken curve by a right action of g_0^{-1}. To get transformed representatives of $G^{\mathbf{C}}/\hat{H}'$, ξ', we need a local $\hat{H}'$ compensator from the right.

are related by a right action of $G^{\mathbf{C}}$ through the relation

$$\xi\vec{v} = \xi g_0^{-1} \cdot g_0 \vec{v} = \left[\xi g_0^{-1} \hat{h}'^{-1}(\xi, g_0)\right] \vec{v}\,' = \xi' \vec{v}\,' \;, \tag{2.7}$$

as

$$\xi' = \xi g_0^{-1} \hat{h}'^{-1}(\xi, g_0), \qquad \hat{h}' \in \hat{H}' \;. \tag{2.8}$$

This relation is sketched in Fig. 1.

We can summarize these facts as follows: Suppose we choose a coordinate system whose origin is a symmetric point $\vec{v}$, and move to a non-symmetric point by an action of g_0 and define a new coordinate system whose origin is $g_0\vec{v}$. Then, unless g_0 belongs to the isometry G of the metric, the origin of the new coordinate system φ' is no longer a symmetric point. The right action (2.8) can be written explicitly as

$$e^{i\varphi' \cdot Z'} = e^{i\varphi \cdot Z} {g_0}^{-1} e^{-iu'(\xi, g_0) \cdot K'}, \qquad \hat{h}' = e^{iu'(\xi, g_0) \cdot K'} \in \hat{H}' \;, \tag{2.9}$$

where u' is a function of g_0 and φ. It can be rewritten as

$$e^{i\varphi' \cdot Z} = {g_0}^{-1} e^{i\varphi \cdot Z} e^{-iu'(\xi, g_0) \cdot K}, \qquad e^{iu'(\xi, g_0) \cdot K} \in \hat{H}, \tag{2.10}$$

and if g_0 is restricted in G, it reduces to the ordinary left action (1.14). The right action does not change the Kähler potential from Eq. (2.7),

$$\vec{v}^\dagger \xi^\dagger \xi \vec{v} \to \vec{v}\,'^\dagger \xi'^\dagger \xi' \vec{v}\,' = \vec{v}^\dagger \xi^\dagger \xi \vec{v}. \tag{2.11}$$

We should again emphasize that this is just a coordinate transformation from the coordinate whose origin is a symmetric point $\vec{v}$ to one whose origin is a non-symmetric point $\vec{v}\,'$, but not a symmetry.

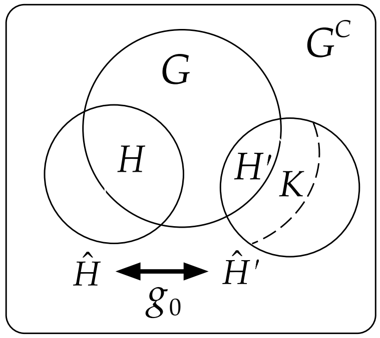

Figure 2

The large circle indicates the group G. The small circles denote the complex subgroups $\hat{H}$ and $\hat{H}'$. $\hat{H}'$ is the transform of $\hat{H}$ by g_0. The real subgroups H or H' are defined as intersections of G and $\hat{H}$ or $\hat{H}'$. K is the image of H by the g_0 transformation. In general H' is a subset of K.

2.2 Supersymmetric vacuum alignment

This subsection is devoted to an another interpretation, in terms of the group theory, about phenomena discussed in the last subsection, and then can be skipped. The compact subgroup G is called a real form of $G^{\mathbf{C}}$. The operation of $\cap\mathcal{G}$ on the complex algebra $\mathcal{G}^{\mathbf{C}}$ or its subalgebra picks up Hermitian generators. The real unbroken symmetry at the vacuum $\vec{v}$ is defined by

$$H = \hat{H} \cap G, \tag{2.12}$$

and at the vacuum $\vec{v}'$ by

$$H' = \hat{H}' \cap G \neq H, \qquad (H') \subset (H), \tag{2.13}$$

where $(\,\cdot\,)$ denotes an equivalence class by the G action: $(H) = \{gHg^{-1} | g \in G\}$. Hence, at the non-symmetric point, the unbroken symmetry group becomes smaller than at the symmetric point [11, 17, 18, 13]. This phenomenon is called the "supersymmetric vacuum alignment". It comes from the different embedding of $\hat{H}$ in $G^{\mathbf{C}}$ as in Fig. 2.[6] In the case of $\hat{H}'$, $K \stackrel{\text{def}}{=} g_0 H g_0^{-1}$ ($\simeq H$) is not a subgroup of G, and the real unbroken symmetry is $K \cap G = H'$ ($\neq H$). At the non-symmetric point, the generators $\mathcal{K} - \mathcal{H}'^{\mathbf{C}}$ ($\cap\ \mathcal{G} = \phi$) are not Hermitian generators. We call them "pseudo-Borel generators".[7] We show in the following sections that they correspond to NG bosons that appear at the non-symmetric point where the unbroken symmetry becomes smaller. The (real) G-orbits of $\vec{v}$ and $\vec{v}'$ are G/H and G/H' ($\neq G/H$),

[6] BKMU called the embedding corresponding to $\vec{v}$ and $\vec{v}'$ the "natural embedding" and the "twisted embedding", respectively [6]. However they discussed the natural embedding only. Kotcheff and Shore called $\vec{v}$ and $\vec{v}'$ the "symmetric embedding" and the "non-symmetric embedding", respectively, since they discussed the case when G/H is a symmetric space and G/H' is a non-symmetric space [11].

[7] A Borel algebra $\mathcal{B}$ ($\in \hat{\mathcal{H}}$) is defined as an algebra which satisfies, $[\mathcal{H}, \mathcal{B}] \subset \mathcal{B}$. Since $\mathcal{K} - \mathcal{H}'^{\mathbf{C}}$ does not satisfy this condition, it is not a Borel subalgebra.

respectively. They are compact submanifolds of $G^{\mathbf{C}}/\hat{H}$ and parametrized by the NG bosons. Other directions of the total target space $G^{\mathbf{C}}/\hat{H}$ are non-compact and correspond to the QNG bosons.

3 NG coset subspaces

In this section, we discuss geometric structures of complex coset manifolds. To be specific we treat an simple example, the $O(N)$ model, but generalizations to other models are straightforward.

3.1 Typical example : $O(N)$ model

In this subsection, we discuss the simplest example, the $O(N)$-model, where the vacuum $\vec{v}$ is in the vector representation of $G = O(N)$: $\vec{v} \in V = \mathbf{R}^N$ [13]. We can complexify the $O(N)$ group by replacing the real vector with the complex vector: $V \to V^{\mathbf{C}} = \mathbf{C}^N$. The generators of the $O(N)$ group are

$$(T_{ij})^k{}_l = \frac{1}{i}(\delta_i{}^k \delta_{jl} - \delta_j{}^k \delta_{il}) = \begin{pmatrix} & & \\ & & i \\ & -i & \\ & & \end{pmatrix} , \qquad (3.1)$$

where only (i,j) and (j,i) elements are non-zero. They satisfy commutation relations and normalization conditions

$$\begin{aligned} &[\, T_{ij}, T_{kl} \,] = -i(\delta_{jk} T_{il} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik} + \delta_{il} T_{jk}), \\ &\mathrm{tr}(T_{ij} T_{kl}) = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned} \qquad (3.2)$$

For later convenience, we define

$$X_i \stackrel{\mathrm{def}}{=} T_{Ni} \ , \ X_i{}' \stackrel{\mathrm{def}}{=} T_{N-1,i'} \ , \ X_{N-1} \stackrel{\mathrm{def}}{=} T_{N,N-1} \ \ (i, i' = 1, \cdots, N-2) \ . \qquad (3.3)$$

Let us classify the real and complex generators at 1) a symmetric point and 2) a non-symmetric point.

1) Symmetric points.
The vacuum vectors of symmetric points can be transformed by a G-action to

$$\vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} . \qquad (3.4)$$

We can immediately find a) the real unbroken algebra and b) the complex unbroken algebra.

1-a) Real unbroken Lie algebra.
Hermitian unbroken generators are $(N-1)\times(N-1)$ matrices which act the first $N-1$ components of (3.4), and others are broken generators:

$$\mathcal{G} = \left(\begin{array}{c|c} \mathcal{H} & * \\ \hline * & 0 \end{array}\right), \tag{3.5}$$

where "$*$" denote Hermitian broken generators. The Hermitian broken generators are of the form

$$X_i = T_{Ni} = \left(\begin{array}{ccc|c} & & & \vdots \\ & \ddots & & i \\ & & & \vdots \\ \hline \cdots & -i & \cdots & 0 \end{array}\right) \in \mathcal{G} - \mathcal{H} \quad (i = 1, \cdots, N-1), \tag{3.6}$$

where dots denote zero components and only i-th elements are nonzero. Thus the symmetry breaking pattern at symmetric points is $G = O(N) \to H = O(N-1)$. A real target manifold, parametrized by NG bosons for this breaking, is a compact homogeneous manifold $O(N)/O(N-1) \simeq S^{N-1}$.

1-b) Complex unbroken Lie algebra.
To discuss the full target manifold, we must discuss by the comlexification of G. By the complexification, however, no new generator appears that leave the vacuum expectation value invariant at the symmetric point. Broken and unbroken generators are simply

$$\begin{cases} Z_R = X_i \in \mathcal{G}^{\mathbf{C}} - \hat{\mathcal{H}}, \\ K_M = H_a \in \hat{\mathcal{H}}. \end{cases} \tag{3.7}$$

All Z_R are Hermitian generators. Chiral superfields, whose bosonic parts are coset coordinates, proportional to Hermitian generators are called the "mixed-type superfield". A real part of a mixed type superfield is a NG boson while an imaginary part corresponds to a *QNG boson*. Since numbers of the NG and QNG bosons are both $N-1$ in this case, this nonlinear realization is called the maximal realization.

2) Non-symmetric points.
We discuss the symmetry breaking in general points. We move the vacuum expectation value $\vec{v}$ to $\vec{v}'$ by the following element of $G^{\mathbf{C}}$:

$$\begin{aligned} g_0 &= \exp(i\theta X_{N-1}) \\ &= \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array}\right) = \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & \cosh\tilde{\theta} & -i\sinh\tilde{\theta} \\ 0 & i\sinh\tilde{\theta} & \cosh\tilde{\theta} \end{array}\right) \in G^{\mathbf{C}}, \end{aligned} \tag{3.8}$$

where $\theta \stackrel{\text{def}}{=} i\tilde{\theta}$ is a pure imaginary angle. Although we have chosen the rotation by X_{N-1}, rotations by other broken generators are equivalent, since they can be transformed by an action of G to each other. Then the vacuum vector at the non-symmetric points can be written, without loss of generality, as

$$\vec{v}' = g_0 \vec{v} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -iv\sinh\tilde{\theta} \\ v\cosh\tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha \\ \beta \end{pmatrix}, \tag{3.9}$$

where we have defined

$$\begin{cases} \beta \stackrel{\text{def}}{=} v\cosh\tilde{\theta} \ : \ \text{real} \\ \alpha \stackrel{\text{def}}{=} -iv\sinh\tilde{\theta} \ : \ \text{pure imaginary}, \end{cases} \tag{3.10}$$

and these satisfy $\beta^2 + \alpha^2 = v^2$. By these constants, g_0 and its inverse can be written as

$$g_0 = \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & \beta/v & \alpha/v \\ 0 & -\alpha/v & \beta/v \end{array}\right), \qquad {g_0}^{-1} = \left(\begin{array}{c|cc} 1 & 0 & \\ \hline & \beta/v & -\alpha/v \\ 0 & \alpha/v & \beta/v \end{array}\right). \tag{3.11}$$

The magnitudes of vacuum expectation values are

$$\vec{v}'^{\dagger}\vec{v}' = \beta^2 - \alpha^2 = \beta^2 + \tilde{\alpha}^2 \stackrel{\text{def}}{=} v'^2 , \tag{3.12}$$

$$\vec{v}'^2 = \vec{v}^2 = \beta^2 + \alpha^2 = \beta^2 - \tilde{\alpha}^2 = v^2 . \tag{3.13}$$

We can find from Eq. (3.9) that the non-compact directions are hyperbolic as in Fig. 3. The full target space is a spheroidal hyperboloid and the compact coset G/H is embedded in the symmetric point. (The compact coset G/H', at non-symmetric point, is discussed in the next subsection.)

Let us discuss real and complex Lie algebras at non-symmetric points.
2-a) Real Lie algebra.
At non-symmetric points, the whole generators can be divided into real unbroken algebra $\mathcal{H}'$ and real broken generators as

$$\mathcal{G} = \left(\begin{array}{c|cc} \mathcal{H}' & * & * \\ \hline * & 0 & * \\ * & * & 0 \end{array}\right), \tag{3.14}$$

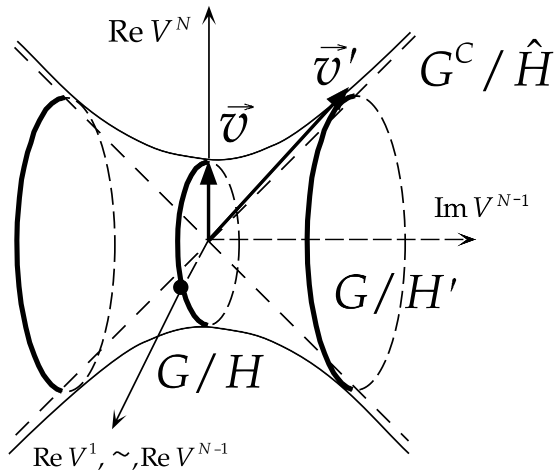

Figure 3

The vertical axis is a real part of V^N, other real parts are written as the axis to this side and the right axis is an imaginary part of V^{N-1}, parametrized by $\tilde{\alpha}$. The NG coset manifold at the symmetric point, G/H, is written as a vertical circle of a radius v, at the center. The NG coset manifold at the non-symmetric point, G/H', is written as two vertical circles of a radius β, and both circles are connected, by a G action, through other imaginary directions of V. (See also Fig. 4, below.)

where "$*$" denote broken generators. The broken generators can be written explicitly as

$$X_i{}' = \left(\begin{array}{ccc|cc} & & & \vdots & \vdots \\ & \ddots & & i & 0 \\ & & & \vdots & \vdots \\ \hline \cdots & -i & \cdots & 0 & 0 \\ \cdots & 0 & \cdots & 0 & 0 \end{array}\right), \qquad X_i = \left(\begin{array}{ccc|cc} & & & \vdots & \vdots \\ & \ddots & & 0 & i \\ & & & \vdots & \vdots \\ \hline \cdots & 0 & \cdots & 0 & 0 \\ \cdots & -i & \cdots & 0 & 0 \end{array}\right),$$

$$X_{N-1} = \left(\begin{array}{c|cc} \ddots & \vdots & \vdots \\ \hline \cdots & 0 & i \\ \cdots & -i & 0 \end{array}\right) \in \mathcal{G} - \mathcal{H}' \qquad (i = 1, \cdots, N-2)\,, \qquad (3.15)$$

where dots denote zero components, and non-zero elements in the first two equations are i-th components. The symmetry breaking pattern at non-symmetric points turns out $G = O(N) \to H' = O(N-2)$, which is smaller than symmetric points. A real target manifold, parametrized by NG bosons, is a compact homogeneous manifold $G/H' = O(N)/O(N-2)$, which is larger than one of the symmetric point, G/H. (See Fig. 3.) Namely we have more NG bosons at non-symmetric points than at symmetric points. These newly emerged NG bosons must come from the QNG

bosons, since the dimension of the full target manifold has to be unchanged. There is only one QNG boson because the number of the NG bosons is $2N-3$ (and the total number is $2N-2$). In the next subsection, we show how these different compact coset manifolds are embedded in the full manifold and how some of the QNG bosons change to the NG bosons at non-symmetric points. Before doing it, we investigate the complex symmetry at non-symmetric points, which give us the key point to understand such phenomena.

2-b) Complex Lie algebra.
Complex broken and unbroken generators at non-symmetric points $\vec{v}'$ can be immediately calculated by using Eqs. (2.4) and (2.3), to yield

$$\begin{aligned} {Z_R}' &= g_0 Z_R g_0^{-1} \\ &= \left\{ \begin{array}{l} g_0 X_i g_0^{-1} = \frac{\alpha}{v} {X_i}' + \frac{\beta}{v} X_i \stackrel{\rm def}{=} {Z_I}' \\ g_0 X_{N-1} g_0^{-1} = X_{N-1} \stackrel{\rm def}{=} Z'_{N-1} \end{array} \right. \in \mathcal{G}^{\bf C} - \hat{\mathcal{H}}' , \end{aligned} \tag{3.16}$$

$$\begin{aligned} {K_M}' &= g_0 K_M g_0^{-1} \\ &= \left\{ \begin{array}{l} g_0 X'_i g_0^{-1} = \frac{\beta}{v} {X_i}' - \frac{\alpha}{v} X_i \stackrel{\rm def}{=} {B_I}' \\ g_0 {H_a}' g_0^{-1} = {H_a}' \in \mathcal{H}' \end{array} \right. \in \hat{\mathcal{H}}' , \end{aligned} \tag{3.17}$$

where ${Z_I}'$ and ${B_I}'$ $(I' = 1, \cdots, N-2)$ can be explicitly written as

$$ {Z_I}' = \left(\begin{array}{ccc|cc} & & & \vdots & \vdots \\ & \ddots & & i\alpha/v & i\beta/v \\ & & & \vdots & \vdots \\ \hline \cdots & -i\alpha/v & \cdots & 0 & 0 \\ \cdots & -i\beta/v & \cdots & 0 & 0 \end{array} \right) , $$

$$ {B_I}' = \left(\begin{array}{ccc|cc} & & & \vdots & \vdots \\ & \ddots & & i\beta/v & -i\alpha/v \\ & & & \vdots & \vdots \\ \hline \cdots & -i\beta/v & \cdots & 0 & 0 \\ \cdots & i\alpha/v & \cdots & 0 & 0 \end{array} \right) . \tag{3.18}$$

We can classify these broken generators to pure-types or mixed-types as follows. First of all, the broken generator ${Z_{N-1}}'$ corresponds to a mixed-type superfield, since ${Z_{N-1}}'$ is a Hermitian generator. A real and a imaginary parts of a scalar component of a chiral superfield generated by ${Z_{N-1}}'$ is a NG boson and a QNG boson, respectively. On the other hand, all other generators ${Z_I}'$ generate pure-type chiral superfields, where both scalar components are NG bosons, since they are non-Hermitian generators.

We can count numbers of the NG and QNG bosons as $2N-3$ and 1, respectively, without using the fact that the total number of the NG and QNG bosons does not change.

3.2 Embedding of NG cosets G/H and G/H'

In this subsection, we show how the different cosets 1) G/H and 2) G/H' are embedded into symmetric and non-symmetric points by using the Shore's procedure [15, 11]. We can obtain coset representatives of NG bosons G/H (G/H') by putting all QNG bosons zero in the complex representative ξ of the full complex coset $G^{\mathbf{C}}/\hat{H}$, at symmetric (non-symmetric) points. In the cases when there are pure-type superfields, we need a local $\hat{H}$-transformation from the right.

1) Embedding of G/H at symmetric points.
Since we do not need a local $\hat{H}$-transformation, we can obtain the representative of G/H by simply putting all QNG bosons zero [11]:

$$\xi|_{\mathrm{QNG}=B^i=0} = e^{i\phi\cdot X} \in G/H \ , \tag{3.19}$$

where fields $\phi = \{A^i\}$ $(i = 1, \cdots, N-1)$ are NG bosons at symmetric points.

2) Embedding of G/H' at non-symmetric points.
The representative of the complex coset at non-symmetric points is

$$\xi(\varphi) = e^{i\varphi\cdot Z'} = \exp i\left[\varphi^i\left(\frac{\beta}{v}X_i + \frac{\alpha}{v}{X_i}'\right) + \varphi^{N-1}X_{N-1}\right] \in G^{\mathbf{C}}/\hat{H}' \ , \tag{3.20}$$

where we have used a character φ as a coordinate. Since there are pure-type broken generators ${Z_I}'$, we need a local $\hat{H}$-transformation

$$\zeta'(\varphi, \varphi^*) = \exp(id(\varphi,\varphi^*)\cdot B') \in \hat{H}' \tag{3.21}$$

from the right:

$$\begin{aligned}
&\xi(\varphi) \to \\
&\hat{\xi}(\hat{A},\hat{B}) \ = \ \xi(\varphi)\zeta'(\varphi,\varphi^*) \\
&\qquad = \ \exp i\left[\hat{\varphi}^i\left(\frac{\beta}{v}X_i + \frac{\alpha}{v}{X_i}'\right) + \hat{d}^i(\hat{\varphi},\hat{\varphi}^*)\left(-\frac{\alpha}{v}X_i + \frac{\beta}{v}{X_i}'\right) + \hat{\varphi}^{N-1}X_{N-1}\right] \\
&\qquad = \ \exp i\left[a^iX_i + b^i{X_i}' \ + (\hat{A}^{N-1} + i\hat{B}^{N-1})X_{N-1}\right] \ ,
\end{aligned} \tag{3.22}$$

where $\hat{\varphi}^i = \hat{A}^i + i\hat{B}^i$ are transformed fields whose relation to φ is obtained below, Eq. (3.28), $\hat{d}^i$ is a function of $\hat{\varphi}$ and $\hat{\varphi}^*$ whose relation to d^i is also obtained below, Eq. (3.27), and a^i and b^i are scalar fields. We can chose the function $\hat{d}$ (or d) such that scalar fields a and b become real:

$$\hat{d}^i(\hat{\varphi},\hat{\varphi}^*) = -\frac{i}{2\tilde{\alpha}\beta}(v'^2\hat{\varphi}^i - v^2\hat{\varphi}^{*i}), \qquad a^i = \hat{A}^i\,\frac{v}{\beta}, \qquad b^i = \hat{B}^i\,\frac{v}{\tilde{\alpha}}. \tag{3.23}$$

Since real scalar fields $\hat{A}^i$ and $\hat{B}^i$ are proportional to Hermitian generators X_i and X_i', respectively in the exponential, they parameterize compact directions of the target manifold. This is why both $\hat{A}^i$ and $\hat{B}^i$ are NG bosons, and $\hat{\Phi}^i$ can be considered

pure-type superfields. On the other hand, since $\hat{A}^{N-1}$ and $\hat{B}^{N-1}$ are proportional to Hermitian and anti-Hermitian generators on the exponential, they parameterize compact and non-compact directions of the target space, respectively. Hence $\hat{A}^{N-1}$ and $\hat{B}^{N-1}$ are NG and QNG bosons, and then $\hat{\Phi}^{N-1}$ is a mixed-type superfield. Then the superfields Φ^i and Φ^{N-1}, before the $\hat{H}$-transformation, turn out to be pure-type and mixed-type superfields, respectively, since they coincide with $\hat{\Phi}^i$ and $\hat{\Phi}^{N-1}$ at the linear order, as shown below, Eq. (3.28).

We can obtain the real representative of the coset G/H' by putting all QNG bosons zero:

$$\hat{\xi}|_{\mathrm{QNG}=\hat{B}^{N-1}=0} = e^{i\phi' \cdot X} \in G/H' \ , \tag{3.24}$$

where fields $\phi' = \{a^i, b^i, \hat{A}^{N-1}\}$ are NG bosons at non-symmetric points. To understand why the compact coset manifold G/H' at non-symmetric points is larger than G/H at symmetric points, see Figs. 4 and 5. At symmetric points, there are $N-1$ non-compact directions, while they change to one non-compact direction and $N-2$ compact directions parametrized by newly emerged NG bosons.[8]

Next we obtain the relation of the fields φ and $\hat{\varphi}$ (or Φ and $\hat{\Phi}$). In general, the first equation of Eq. (3.22) can be written explicitly as

$$e^{i\varphi\cdot Z} e^{id(\varphi,\varphi^*)\cdot K} = e^{i(\hat{\varphi}\cdot Z + \hat{d}(\hat{\varphi},\hat{\varphi}^*)\cdot K)}, \qquad \zeta'(\varphi,\varphi^*) = e^{id(\varphi,\varphi^*)\cdot K} \in \hat{H}' \ , \tag{3.25}$$

where d is a function of φ and φ^*. By using the Baker-Campbell-Hausdorff formula on the left-side, we obtain relations

$$\hat{\varphi}^R = \varphi^R + \frac{1}{2} f_{MS}{}^R \varphi^S d^M + \cdots, \tag{3.26}$$

$$\hat{d}^M = d^M + \frac{1}{12} f_{NR}{}^T f_{TS}{}^M \varphi^R \varphi^S d^N + \cdots. \tag{3.27}$$

Note that we have used only the fact that $G^{\mathbf{C}}/\hat{H}$ is a symmetric space, and the result is model-independent. By these equations, two coordinates $\hat{\varphi}$ and φ are related by

$$\hat{\varphi} = \varphi + O(\varphi^2, \varphi\varphi^*), \tag{3.28}$$

and coincide to each other at the first order.[9] This is why Φ and $\hat{\Phi}$ coincide to each other at the linear level and their identifications to pure- or mixed-type superfields coincide.

[8] We give a comment on the NG submanifold at non-symmetric points, G/H'. It can be considered as a $H/H' \simeq S^{N-2}$ fiber bundle over a base manifold, $G/H \simeq S^{N-1}$. By bringing $\vec{v}'$ to $\vec{v}$, the fiber shrinks but the base remains at finite size (radius v).

[9] Note that this transformation is not holomorphic and so the complex structures of the two coordinates φ and $\hat{\varphi}$ are different. For our purpose to calculate low-energy theorems, the difference can be neglected because the curvature tensors in both coordinates coincide at $\varphi = 0$.

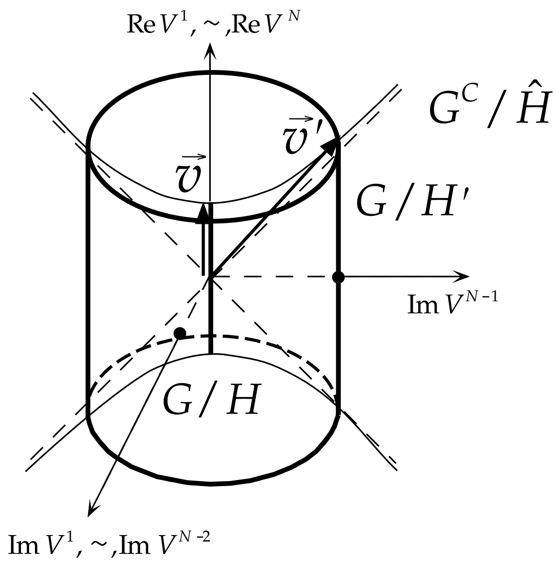

Figure 4

In Fig. 4, the imaginary directions of V are written as a horizontal plane, and real directions are written as a vertical line. G/H is written as a segment at the center and G/H' is written as a cylinder enclosing the G/H. Newly emerged NG bosons are written as a horizontal circle of a radius $\tilde{\alpha}$, which is just a H-orbit of the non-symmetric vacuum $\vec{v}\,'$.

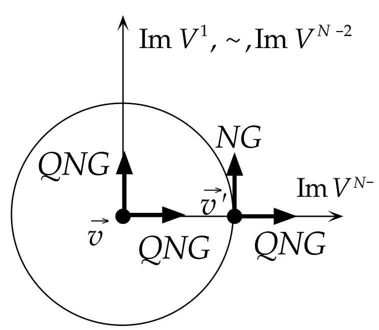

Figure 5

View from the top of Fig. 4. It can be seen that most of QNG bosons at the symmetric point change to newly emerged NG bosons at the non-symmetric point, with the total number of massless bosons been unchanged.

4 Low-energy theorems at general points

In this section, we discuss low-energy theorems at non-symmetric points. We elaborate on the $O(N)$ model as an example, but generalizations to other models are straightforward.

4.1 Some formula for the $O(N)$ model

Before discussing low-energy theorems, we give comments for a "linear" description of the model. The invariant Lagrangian can be written as

$$\mathcal{L} = \int d^4\theta \vec{\phi}^\dagger \vec{\phi} + \left(\int d^2\theta W(\phi) + (\text{conj.}) \right) , \tag{4.1}$$

where $\vec{\phi}(x, \theta, \bar{\theta})$ consists of chiral superfields belonging to a linear representation of G. $W(\phi)$ is a G-invariant superpotential, and it is actually $G^{\mathbf{C}}$-invariant due to its holomorphy. For the $O(N)$ model, its candidate is

$$W(\phi) = g\phi_0(\vec{\phi}^2 - a^2) , \tag{4.2}$$

where ϕ_0 is an additional G-singlet field and g is a coupling constant. ϕ_0 tends to a non-dynamical auxiliary field in the heavy mass limit ($g \to \infty$). In this limit we can eliminate ϕ_0 by its equation of motion, which gives an F-term constraint among dynamical fields ϕ^i. For the $O(N)$ model, the constraint is $\vec{\phi}^2 - a^2 = 0$. The Kähler potential may suffer from a quantum correction, with preserving the global symmetry G. We thus obtain a nonlinear Kähler potential for NG chiral superfields,

$$K(\Phi, \Phi^\dagger) = f(\vec{\phi}^\dagger \vec{\phi})|_{\mathrm{F}} = f(\vec{v}^\dagger \xi^\dagger \xi \vec{v}) , \tag{4.3}$$

where F denotes an F-term constraint. We have used a relation between linear superfields and NG superfields, $\vec{\phi}|_{\mathrm{F}} = \xi\vec{v}$. This recovers Eq. (1.15).

If we restrict the problem to the $O(N)$ model, geometric quantities can be calculated by solving the constraint $\vec{\phi}^2 = a^2$ explicitly as $\phi^N = \sqrt{a^2 - \sum_{i=1}^{N-1} (\phi^i)^2}$. However we discuss in the coset formalism which can be generalized to other models straightforwardly.

To obtain geometric quantities in the coset formalism, we need expectation values of broken generators, sandwiched by the vacuum vector $\vec{v}'$. Let us calculate them first. We use indices $R, S, \cdots = 1, \cdots, N-1$ and $I, J, \cdots = 1, \cdots, N-2$. We omit primes except for $\vec{v}'$. By noting that ${Z_{N-1}}^\dagger = Z_{N-1}$ and ${Z_I}^\dagger \neq Z_I$, we calculate products of one complex generator on vacuum expectation values, given by

$$Z_I \, \vec{v}' = \begin{pmatrix} 0 \\ \vdots \\ iv \\ \vdots \\ 0 \end{pmatrix} , \qquad Z_{N-1} \, \vec{v}' = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ i\beta \\ -i\alpha \end{pmatrix} , \qquad {Z_I}^\dagger \, \vec{v}' = \frac{v'^2}{v^2} \begin{pmatrix} 0 \\ \vdots \\ iv \\ \vdots \\ 0 \end{pmatrix} , \tag{4.4}$$

where the I-th elements are nonzero in the first and the third equations. Products of two complex generators on vacuum expectation values are also given by

$$Z_R Z_S\,\vec{v}\,' = \delta_{RS}\vec{v}\,', \qquad {Z_I}^\dagger Z_J\,\vec{v}\,' = \delta_{I^*J}\vec{v}\,'^*\ ,$$
$${Z_I}^\dagger Z_{N-1}\,\vec{v}\,' = \frac{c^2}{v'^2}{Z_I}^\dagger\,\vec{v}\,', \qquad Z_{N-1}{Z_I}^\dagger\,\vec{v}\,' = 0\ , \tag{4.5}$$

where we have defined $c^2 \stackrel{\text{def}}{=} 2\sqrt{v'^4 - v^4} = 2\tilde{\alpha}\beta$. We define convenient notations

$$\langle R\cdots\rangle \stackrel{\text{def}}{=} \vec{v}\,'^\dagger Z_R\cdots\vec{v}\,', \qquad R^\dagger \stackrel{\text{def}}{=} {Z_R}^\dagger. \tag{4.6}$$

Then expectation values of one to four generators can be calculated, to yield

$$\begin{aligned}
&\langle I\rangle = 0, && \langle N-1\rangle = c^2\ ,\\
&\langle RS\rangle = v'^2\delta_{RS}, && \langle I^\dagger J\rangle = v^2\delta_{I^*J}\ ,\\
&\langle IJK\rangle = 0, && \langle I^\dagger JK\rangle = 0\ ,\\
&\langle N-1, IJ\rangle = c^2\delta_{IJ}\ , && \langle IJ, N-1\rangle = 0\ ,\\
&\langle *\,I, N-1\rangle = 0\ , && \langle *, N-1, I\rangle = 0\ ,\\
&\langle I, N-1, N-1\rangle = 0\ , && \langle I^\dagger, N-1, N-1\rangle = 0\ ,\\
&\langle N-1, I, N-1\rangle = 0\ , && \langle N-1, N-1, N-1\rangle = c^2\ ,\\
&\langle R^\dagger S^\dagger UV\rangle = v'^2\delta_{R^*S^*}\delta_{UV}\ ,
\end{aligned} \tag{4.7}$$

where "$*$" denotes an arbitrary generator. These quantities are needed for the calculation of the curvature tensor. They can be generalized to other models straightforwardly.

4.2 Geometric quantities and low-energy theorems

We can calculate geometric quantities of the $O(N)$ model by using the formulas obtained in the last subsection. In this section, we consider the most simple Kähler potential, $K = f(x) = x$. The general case is discussed in Appendix B.

First of all the metric is given by (we omit prime on Z_R)

$$g_{ij^*} = \partial_i\partial_{j^*}K = G_{RS^*}E_i^R(E_j^S)^*, \qquad G_{RS^*} = \vec{v}\,'^\dagger {Z_S}^\dagger\xi^\dagger\xi Z_R\vec{v}\,', \tag{4.8}$$

where G_{RS^*} is called an auxiliary metric. The auxiliary metric at the point $\varphi = 0$ becomes

$$G_{RS^*}|_{\varphi=0} = \langle S^\dagger R\rangle = \begin{pmatrix} v^2\delta_{IJ^*} & 0\\ 0 & v'^2\end{pmatrix}\ . \tag{4.9}$$

A vielbein and a $\hat{H}$-connection at the point $\varphi = 0$ are given by

$$E_i^R|_{\varphi=0} = \delta_i^R\ , \qquad W_i^M|_{\varphi=0} = 0, \tag{4.10}$$

respectively, and the differentiation of the vielbein with respect to the coordinate can be calculated, to yield

$$\partial_j E_i^R|_{\varphi=0} = 0 \ . \tag{4.11}$$

Let us calculate a complex curvature

$$R_{ij^*kl^*} = \partial_i\partial_{j^*}\partial_k\partial_{l^*}K - g^{mn^*}(\partial_{j^*}\partial_m\partial_{l^*}K)(\partial_i\partial_k\partial_{n^*}K) \ , \tag{4.12}$$

which is crucial to low-energy theorems. The complex curvature on the origin $\varphi = 0$ of $G^{\mathbf{C}}/\hat{H}$ can be calculated by Eq. (4.7), to yield

$$\begin{aligned} R_{ij^*kl^*}|_{\varphi=0} &= [\langle S^\dagger V^\dagger U R\rangle - G^{XY^*}|_{\varphi=0}\langle X^\dagger S V\rangle^* \langle Y^\dagger R U\rangle]\delta_i^R(\delta_j^S)^*\delta_k^U(\delta_l^V)^* \\ &= \frac{v^4}{v'^2}\delta_{ik}\delta_{j^*l^*} \ . \end{aligned} \tag{4.13}$$

The first line is for general symmetric manifolds $G^{\mathbf{C}}/\hat{H}$ with one vacuum expectation value, and the second line is for the $O(N)$ model. If we rescale fields so that the metric (4.9) becomes the Knonecker's delta, components of the curvature tensor become

$$R_{ij^*kl^*}|_{\varphi=0} = \begin{cases} 1/v'^2\delta_{ik}\delta_{j^*l^*} & \text{when } i,j,k,l = 1,\cdots,N-2, \\ v^2/v'^4\delta_{ik}\delta_{j^*l^*} & \text{when only two indices are } N-1\text{-th}, \\ v^4/v'^6\delta_{ik}\delta_{j^*l^*} & \text{when } i,j,k,l = N-1. \end{cases} \tag{4.14}$$

From these equations, we can calculate real components of the curvature in the rescaled coordinate given by $(i,j,k,l = 1,\cdots,N-2)$

$$\begin{aligned} &R_{A^iA^jA^kA^l} = 2\frac{1}{v'^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\ &R_{B^iA^jB^kA^l} = -2\frac{1}{v'^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ &R_{B^iA^jA^kA^l} = R_{A^iA^jB^kA^l} = 0, \\ &R_{A^{N-1}A^jA^{N-1}A^l} = 2\frac{v^2}{v'^4}\delta_{jl}, \\ &R_{A^{N-1}A^{N-1}A^kA^l} = 0, \\ &R_{B^{N-1}A^jB^{N-1}A^l} = -2\frac{v^2}{v'^4}\delta_{jl}, \\ &R_{B^{N-1}A^{N-1}B^kA^l} = 0, \\ &R_{B^{N-1}A^{N-1}B^{N-1}A^{N-1}} = -4\frac{v^4}{v'^6} \ , \end{aligned} \tag{4.15}$$

where all quantities are evaluated at $\varphi = 0$.

Let us discuss low-energy theorems. As discussed in Appendix A, components of the curvature tensor in an arbitrary coordinate and in normal coordinates coincide to each other in this order. Hence, by substituting these equations to Eq. (1.7), we

can obtain low-energy theorems for two-body scattering amplitudes among NG and QNG bosons at general points of target spaces.

At the symmetric point, $v' = v$, all coefficients become $\frac{1}{v^2}$. There, all A^i and A^{N-1} fields are NG bosons of the symmetry breaking, $O(N)$ to $O(N-1)$, and we can verify that their scattering amplitudes satisfy low-energy theorems from a equation, ${f_{ij}}^a f_{akl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. Their decay constant is $f_\pi = v$. All B fields correspond to QNG bosons and their low-energy theorems coincide with those of NG partners as discussed in Sec. 1 and Ref [14].

At the non-symmetric point, there appear new features of low-energy theorems for NG and QNG bosons. The unbroken symmetry $O(N-1)$ at symmetric points further breaks down to $O(N-2)$, and B^i $(i = 1, \cdots, N-2)$ change to NG bosons for the second breaking of $O(N-1)$ to $O(N-2)$. From the first equation of (4.15) and $R_{B^iB^jB^kB^l} = R_{A^iA^jA^kA^l}$, we can verify that low-energy theorems for $B^i (i = 1, \cdots, N-2)$ coincide with those of NG bosons of the second symmetry breaking, $O(N-1)$ to $O(N-2)$. Low-energy theorems among A^i and A^N (at symmetric points) for the first breaking of $O(N)$ to $O(N-1)$ are distorted there, since the field A^{N-1} becomes to play a special role.

Before closing this section, we give a comment on a relation between NG bosons at non-symmetric points in a supersymmetric theory and NG bosons in a non-supersymmetric theory. In a non-supersymmetric theory with spontaneously broken $O(N)$ symmetry to a subgroup $O(N-2)$, two sets of linear fields $\vec{\phi}_1$ and $\vec{\phi}_2$, belonging to the vector representation, should have vacuum expectation values. The broken generators are

$$\mathcal{G} - \mathcal{H} = \left(\begin{array}{c|c|c} 0 & f_\pi^{(1)} & f_\pi^{(2)} \\ \hline f_\pi^{(1)} & & f_\pi^{(3)} \\ \hline f_\pi^{(2)} & f_\pi^{(3)} & \end{array} \right), \tag{4.16}$$

where we have expressed three H-irreducible sectors of broken generators by decay constants $f_\pi^{(i)}$ $(i = 1, 2, 3)$ of NG bosons corresponding to these generators.[10] These three free parameters in the non-supersymmetric theory are reduced to two parameters, v and v', to be embedded into a bosonic part of a supersymmetric theory. This is because there exist $N-2$ pure-type multiplets and they relate two decay constants, $f_\pi^{(1)}$ and $f_\pi^{(2)}$, in Eq. (4.16). This was known at least in pure-realization cases, where there exist only pure-type multiplets [8].

[10] Since there are two vacuum expectation values $\vec{v}_1 = \langle \vec{\phi}_1 \rangle$, $\vec{v}_2 = \langle \vec{\phi}_2 \rangle \in \mathbf{N}$ to break $O(N)$ to $O(N-2)$, these three sectors correspond to three G-invariants, $\vec{v}_1^{\,2}$, $\vec{v}_2^{\,2}$, $\vec{v}_1 \cdot \vec{v}_2$.

5 Conclusion and discussion

If a global symmetry spontaneously breaks in supersymmetric theories, there appear NG and QNG bosons and their fermions superpartners. The low-energy effective Lagrangian for these fields can be constructed as supersymmetric nonlinear sigma models. If symmetry breaking occur by a superpotential of a fundamental of an effective field theories, there must appear at least one QNG boson. Hence the target manifold inevitably becomes non-compact. As a result, supersymmrtic vacuum alignment occur; NG and QNG boson can change with the total number preserved. This has been understood by different embedding of $\hat{H}$ into $G^{\mathbf{C}}$. Low-energy theorems of two-body scattering amplitudes for these bosons was known at symmetric points.

In this paper, we have calculated low-energy theorems for NG and QNG bosons at general points. We have found new features of low-energy theorems. In a theory with the supersymmetric vacuum alignment, symmetry breaking occurs twice (or more times for other models). The low-energy theorems for the first breaking at symmetric points have been distorted at non-symmetric points; on the other hand, low-energy theorems for the second breaking coincide with non-supersymmetric cases. This is because one (or some for other models) NG boson must sit in mixed-type multiplet, and play a special role in the sense that its partner is QNG boson.

Although we have illustrated the low-energy theorems at non-symmetric points in the $O(N)$-model with the simplest (linear) Kähler potential, generalizations to more complicated models are straightforward. (The calculation in the most general Kähler potential of the $O(N)$-model is discussed in Appendix B.) Consider the case that there are n $G^{\mathbf{C}}$-invariants and m G-invariants. (In the $O(N)$-model, they are $m = n = 1$, since there is one $G^{\mathbf{C}}$-invariant, $\vec{\phi}^2$, and one G-invariant, $|\vec{\phi}|^2$.) The low-energy effective Kähler potential can be written as an arbitrary function of m G-invariants [13]. We can count parameters included in low-energy theorems of two-body scattering amplitudes. Since curvature tensor includes from one to four derivatives of the arbitrary function, there are certain numbers of the parameters concerned with the arbitrary function. Therefore the low-energy theorems include these parameters. As seen in this paper, in the case of $O(N)$ model, there were six parameters $v, v', f_1, f_2, f_3, f_4$ (see Appendix B).

Although we have investigated two-body scattering amplitudes, a generalization to many-body scattering amplitudes can be calculated by using the Kähler normal coordinate to the desired order [19]. An interaction Lagrangian can be written by the curvature tensor, covariant derivatives of the curvature tensor etc. If we calculate n-body scattering amplitudes, it contains from one to n derivatives of the arbitrary function.

We have investigated only low-energy theorems, namely low-energy scattering amplitudes at the leading order $\mathcal{O}(p^2)$. It is interesting, for a development of supersymmetric chiral perturbation theories, to investigate higher derivative terms such as next-to leading terms $\mathcal{O}(p^4)$ [20]. At such order, we need a supersymmetric

Wess-Zumino-Witten term [21], which correctly reproduces anomalies of the global symmetry (we did not need it at the lowest order).

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A Kähler normal coordinate expansion

In this appendix, we show that the Kähler normal coordinate can be always obtained from general coordinates by a holomorphic coordinate transformation preserving the curvature tensor up to a constant order. The systematic method to obtain the Kähler normal coordinate to an arbitrary order is discussed in Ref. [19].

Let $\{z^i, z^{*i}\}$ are the general coordinate. We expand the Kähler potential by the Taylor expansion as

$$\begin{aligned} K(z,z^*) &= K|_0 + F(z) + F^*(z^*) \\ &\quad + g_{ij^*}|_0\, z^i z^{*j} + \frac{1}{2}\Gamma_{i^*jk}|_0\, z^{*i} z^j z^k + \frac{1}{2}\Gamma_{ij^*k^*}|_0\, z^i z^{*j} z^{*k} \\ &\quad + \frac{1}{4}(R_{ij^*kl^*} + g_{mn^*}\Gamma^m{}_{ik}\Gamma^{n^*}{}_{j^*l^*})|_0\, z^i z^k z^{*j} z^{*l} \\ &\quad + \frac{1}{6}\partial_k \Gamma_{l^*ij}|_0\, z^i z^j z^k z^{*l} + \frac{1}{6}\partial_{k^*}\Gamma_{li^*j^*}|_0\, z^{*i} z^{*j} z^{*k} z^l + O(z^5), \end{aligned} \tag{A.1}$$

where

$$F(z) = \partial_i K|\, z^i + \frac{1}{2}\partial_i \partial_j K|\, z^i z^j + \cdots \tag{A.2}$$

is holomorphic and can be eliminated by a Kähler transformation. Here $\Gamma^i{}_{jk}$ is the connection and $R_{ij^*kl^*}$ is the curvature tensor of the Kähler manifold [22]. There are many non-covariant coefficients except for g_{ij^*} and $R_{i^*jk^*l}$. To eliminate them note that Eq. (A.1) can be written as

$$\begin{aligned} &K(z,z^*) \\ &= K|_0 + F(z) + F^*(z^*) \\ &+ g_{mn^*}(z^m + \frac{1}{2}\Gamma^i{}_{jk}|z^j z^k + \frac{1}{6}\partial_k \Gamma^m{}_{ij}|z^i z^j z^k)(z^n + \frac{1}{2}\Gamma^i{}_{jk}|z^j z^k + \frac{1}{6}\partial_k \Gamma^n{}_{ij}|z^i z^j z^k)^* \\ &+ \frac{1}{4}R_{i^*jk^*l}|z^{*i} z^{*k} z^j z^l + O(z^5). \end{aligned} \tag{A.3}$$

By a holomorphic coordinate transformation

$$\omega^i = z^i + \frac{1}{2}\Gamma^i{}_{jk}|z^j z^k + \frac{1}{6}\partial_l \Gamma^i{}_{jk}|z^j z^k z^l + O(z^4), \tag{A.4}$$

it can be rewritten as

$$K(\omega,\omega^*) = K| + \tilde{F}(\omega) + \tilde{F}^*(\omega^*) + g_{ij^*}|\omega^i\omega^{*j} + \frac{1}{4}R_{i^*jk^*l}|\omega^{*i}\omega^{*k}\omega^j\omega^l + O(\omega^5), \quad \text{(A.5)}$$

where $\tilde{F}(\omega) \stackrel{\text{def}}{=} F(z(\omega))$. The new coordinate ω is the Kähler normal coordinate to the forth order. In this coordinate, all coefficients are covariant quantities.

By performing these transformations in the superfield level, we obtain a Kähler normal coordinate expansion of the Lagrangian given by [19]

$$\begin{aligned}
\mathcal{L} &= g_{ij^*}|_{\varphi=0}\,\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + ig_{ij^*}|_{\varphi=0}\,\bar{\psi}^j\bar{\sigma}^\mu\partial_\mu\psi^i \\
&+ R_{ij^*kl^*}|_{\varphi=0}\,\varphi^k\varphi^{*l}\partial_\mu\varphi^i\partial^\mu\varphi^{*j} + \frac{1}{4}R_{ij^*kl^*}|_{\varphi=0}\,\psi^i\psi^k\bar{\psi}^j\bar{\psi}^l \\
&+ iR_{ij^*kl^*}|_{\varphi=0}\,\varphi^{*j}\partial_\mu\varphi^i(\bar{\psi}^l\bar{\sigma}^\mu\psi^k). \qquad \text{(A.6)}
\end{aligned}$$

which has been used to calculate the low-energy theorems. First two terms are equation terms for bosons and fermions, and others can be considered interaction terms. (Although we concentrate on bosonic amplitudes in this paper, we can also obtain low-energy theorems including fermion partners of NG and QNG bosons by using this expansion.)

Next let us discuss a relation between the curvature tensor in an arbitrary coordinate and in the Kähler normal coordinate. Since the Jacobian of (A.4)

$$J^i{}_j = \partial\omega^i/\partial z^j = \delta^i_j + O(z) \qquad \text{(A.7)}$$

is unit matrix up to constant order, components of the curvature tensor in the new coordinate ω is

$$R'_{m^*no^*p} = R_{i^*jk^*l}(J^i{}_m)^*J^j{}_n(J^k{}_o)^*J^l{}_p = R_{i^*jk^*l} + O(\omega)\ . \qquad \text{(A.8)}$$

Therefore components of the curvature tensor is invariant up to constant order. We can use an arbitrary coordinate to calculate the curvature tensor in low-energy theorems (1.7), although low-energy theorems theirselves have been obtained in normal coordinates.

B General Kähler potential

In Sec. 4, we have calculated geometric quantities in the case of the simplest Kähler potential $K = f(x) = x$. In this appendix we calculate them in an arbitrary case $K = f(x)$.

We can calculate the geometric quantities of the $O(N)$ model by using the formulas obtained in Sec. 4.1. First the metric is (we omit primes except for $\vec{v}'$)

$$\begin{aligned}
&g_{ij^*} = \partial_i\partial_{j^*}K = G_{RS}E^R_i(E^S_j)^*, \\
&G_{RS^*} = f'(z)(\vec{v}'^\dagger Z_S{}^\dagger\xi^\dagger\xi Z_R\vec{v}') + f''(z)(\vec{v}'^\dagger Z_S{}^\dagger\xi^\dagger\xi\vec{v}')(\vec{v}'^\dagger\xi^\dagger\xi Z_R\vec{v}'), \\
&z \stackrel{\text{def}}{=} \vec{v}'^\dagger\xi^\dagger\xi\vec{v}'. \qquad \text{(B.1)}
\end{aligned}$$

At the point $\varphi = 0$, we define derivatives of the arbitrary function by

$$f_1 \stackrel{\rm def}{=} f'(\vec{v}'^\dagger \vec{v}') = f'(v'^2), \qquad f_2 \stackrel{\rm def}{=} f''(\vec{v}'^\dagger \vec{v}') = f''(v'^2) \ , \ \cdots \ . \tag{B.2}$$

The auxiliary metric at the point $\varphi = 0$ is

$$G_{RS^*}|_{\varphi=0} = f_1 \langle S^\dagger R \rangle + f_2 \langle S^\dagger \rangle \langle R \rangle \ , \tag{B.3}$$

where we have used the notations in Eq. (4.6). This becomes for the $O(N)$ model

$$G_{RS^*}|_{\varphi=0} = \begin{pmatrix} f_1 v^2 \delta_{IJ^*} & 0 \\ 0 & f_1 v'^2 + f_2 c^4 \end{pmatrix} \ , \tag{B.4}$$

from Eq. (4.7). The vielbein and the $\hat{H}$-connection at the point $\varphi = 0$ are

$$E_i^R|_{\varphi=0} = \delta_i^R, \qquad W_i^M|_{\varphi=0} = 0, \tag{B.5}$$

respectively, and differentiations of the vielbein with respect to coordinates are

$$\partial_j E_i^R|_{\varphi=0} = 0 \ . \tag{B.6}$$

The curvature tensor (4.12) on the point $\varphi = 0$ of an arbitrary symmetric $G^{\mathbf{C}}/\hat{H}$ is given by

$$\begin{aligned}
& R_{RS^*UV^*}|_{\varphi=0} \\
&= f_1 \langle S^\dagger V^\dagger U R \rangle + f_2 (\langle V^\dagger U \rangle \langle S^\dagger R \rangle + \langle S^\dagger U \rangle \langle V^\dagger R \rangle + \langle S^\dagger V^\dagger \rangle \langle U R \rangle \\
&\qquad + \langle V^\dagger \rangle \langle S^\dagger U R \rangle + \langle U \rangle \langle S^\dagger V^\dagger R \rangle + \langle S^\dagger V^\dagger U \rangle \langle R \rangle + \langle S^\dagger \rangle \langle V^\dagger U R \rangle) \\
&\quad + f_3 (\langle V^\dagger \rangle \langle U \rangle \langle S^\dagger R \rangle + \langle V^\dagger \rangle \langle S^\dagger U \rangle \langle R \rangle + \langle V^\dagger \rangle \langle S^\dagger \rangle \langle U R \rangle \\
&\qquad + \langle V^\dagger U \rangle \langle S^\dagger \rangle \langle R \rangle + \langle U \rangle \langle S^\dagger V^\dagger \rangle \langle R \rangle + \langle U \rangle \langle S^\dagger \rangle \langle V^\dagger R \rangle) + f_4 \langle V^\dagger \rangle \langle U \rangle \langle S^\dagger \rangle \langle R \rangle \\
&\quad - G^{XY^*}|_{\varphi=0} \\
&\quad \times (f_1 \langle X^\dagger S V \rangle + f_2 (\langle S \rangle \langle X^\dagger V \rangle + \langle X^\dagger S \rangle \langle V \rangle + \langle X^\dagger \rangle \langle S V \rangle) + f_3 \langle S \rangle \langle X^\dagger \rangle \langle V \rangle)^* \\
&\quad \times (f_1 \langle Y^\dagger R U \rangle + f_2 (\langle R \rangle \langle Y^\dagger U \rangle + \langle Y^\dagger R \rangle \langle U \rangle + \langle Y^\dagger \rangle \langle R U \rangle) + f_3 \langle R \rangle \langle Y^\dagger \rangle \langle U \rangle),
\end{aligned} \tag{B.7}$$

where have defined

$$R_{ij^*kl^*} = R_{RS^*UV^*} \delta_i^R (\delta_j^S)^* \delta_k^U (\delta_l^V)^* \ . \tag{B.8}$$

In the case of the $O(N)$ model, it can be calculated from Eq. (4.7), to yield

$$\begin{aligned}
R_{IJ^*KL^*} &= v^4 \left[\frac{{f_1}^2 + f_1 f_2 v'^2}{f_1 v'^2 + f_2 c^4} \right] \delta_{IK} \delta_{J^*L^*} + f_2 v^4 (\delta_{IJ^*} \delta_{KL^*} + \delta_{IL^*} \delta_{KJ^*}) \ , \\
R_{N-1,J^*,KL^*} &= R_{I,N-1,KL^*} = 0 \ , \\
R_{N-1,N-1^*,KL^*} &= v^2 \left[\frac{f_1 f_2 v'^2 + (f_1 f_3 - {f_2}^2) c^4}{f_1} \right] \delta_{KL^*} \ ,
\end{aligned}$$

$$
\begin{aligned}
&R_{N-1,J^*,N-1,L^*} = v^4 \left[\frac{{f_1}^2 + f_1 f_2 v'^2 - (2{f_2}^2 + f_1 f_3)c^4}{f_1 v'^2 + f_2 c^4}\right] \delta_{J^*L^*} \ , \\
&R_{N-1,N-1^*,N-1,L^*} = R_{N-1,N-1^*,K,N-1^*} = 0 \ , \\
&R_{N-1,N-1^*,N-1,N-1^*} \\
&= \frac{1}{f_1 v'^2 + f_2 c^4}\Big[{f_1}^2 v^4 + f_1 f_2 v'^2 (2v'^4 + v^4) - 2{f_2}^2 (v'^8 + v^4 v'^4 - 2v^8) \\
&\qquad +2 f_1 f_3 c^4 (2v'^4 + v^4) + f_1 f_4 c^8 v'^2 + (f_2 f_4 - {f_3}^2) c^{12}\Big], \qquad \text{(B.9)}
\end{aligned}
$$

where $c^4 = v'^4 - v^4$. We can show that these results coincide with direct calculations after solving the constraint $\vec{\phi}^2 = a^2$ as $\phi^N = \sqrt{a^2 - \sum_{i=1}^{N-1}(\phi^i)^2}$. However the coset formalism presented in this paper can be generalized to an arbitrary model straightforwardly.

Before the calculation of real components of the curvature tensor, we give comments.

1. At the symmetric point, $v'^2 = v^2,\ c^2 = \sqrt{v'^4 - v^4} = 0$, the curvatures can be written as

$$
R_{ij^*kl^*} = \frac{1}{2}{f_\pi}^2 \delta_{ik}\delta_{j^*l^*} + g^2(\delta_{ik}\delta_{j^*l^*} + \delta_{ij^*}\delta_{kl^*} + \delta_{il^*}\delta_{kj^*}) \ , \qquad \text{(B.10)}
$$

$$
{f_\pi}^2 = 2 f_1 v^2 \ , \qquad g^2 = f_2 v^4. \qquad \text{(B.11)}
$$

We recovered previous results in Ref. [14] and Eq. (1.19) for the $O(N)$ model.

2. For the case of the linear Kähler potential,

$$
f(x) = x \quad : \quad f_1 = 1 \ , \ f_2 = f_3 = \cdots = 0 \ , \qquad \text{(B.12)}
$$

they reduce to results in Eq. (4.13).

We can calculate real components of the curvature tensor which are directly concerned with low-energy theorems, given by $(i,j,k,l = 1,\cdots,N-2)$

$$
\begin{aligned}
&R_{A^iA^jA^kA^l} = 2v^4 \left[\frac{{f_1}^2 - {f_2}^2 c^4}{f_1 v'^2 + f_2 c^4}\right] (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\
&R_{B^iA^jB^kA^l} = -2v^4 \left[\frac{{f_1}^2 + 2 f_1 f_2 v'^2 + {f_2}^2 c^4}{f_1 v'^2 + f_2 c^4}\right] (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - 4 f_2 v^4 (\delta_{ij}\delta_{kl}), \\
&R_{B^iA^jA^kA^l} = R_{A^iA^jB^kA^l} = 0, \\
&R_{A^{N-1}A^jA^{N-1}A^l} \\
&= \frac{2v^2}{f_1(f_1 v'^2 + f_2 c^4)}\Big[{f_1}^3 v^2 + {f_1}^2 f_2 v'^2 (v^2 - v'^2) - 2 f_1 {f_2}^2 v^2 c^4 + {f_2}^3 c^8, \\
&\qquad -{f_1}^2 f_3 (v^2 + v'^2) c^4 - f_1 f_2 f_3 c^8\Big]\delta_{jl}, \\
&R_{A^{N-1}A^{N-1}A^kA^l} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&R_{B^{N-1}A^jB^{N-1}A^l} \\
&= \frac{2v^2}{f_1(f_1v'^2+f_2c^4)}\Big[-{f_1}^3v^2-{f_1}^2f_2v'^2(v^2+v'^2)+2f_1{f_2}^2v^2c^4+{f_2}^3c^8 \\
&\qquad\qquad\qquad\qquad +{f_1}^2f_3(v^2-v'^2)c^4-f_1f_2f_3c^8\Big]\delta_{jl}, \\
&R_{B^{N-1}A^{N-1}B^kA^l} = -4v^2\left[\frac{f_1f_2v'^2+(f_1f_3-{f_2}^2)c^4}{f_1}\right]\delta_{kl}, \\
&R_{B^{N-1}A^{N-1}B^{N-1}A^{N-1}} \\
&= \frac{-4}{f_1v'^2+f_2c^4}\Big[{f_1}^2v^4+f_1f_2v'^2(2v'^4+v^4)-2{f_2}^2(v'^8+v^4v'^4-2v^8) \\
&\qquad\qquad +2f_1f_3c^4(2v'^4+v^4)+f_1f_4c^8v'^2+(f_2f_4-{f_3}^2)c^{12}\Big].
\end{aligned}
\tag{B.13}
$$

At the symmetric point, these reduce to $(i,j,k,l=1,\cdots,N-1)$

$$
\begin{aligned}
R_{A^iA^jA^kA^l} &= {f_\pi}^2(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}), \\
R_{B^iA^jB^kA^l} &= -{f_\pi}^2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-4g^2(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}+\delta_{ij}\delta_{kl}), \\
R_{B^iA^jA^kA^l} &= R_{A^iA^jB^kA^l} = 0\ ,
\end{aligned}
\tag{B.14}
$$

which again coincide with those obtained in Ref. [14] and Eq. (1.22) for the $O(N)$ model.

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